ex: $0.\overline{23} = ?$

$$0.\overline{23} = 0.232323\ldots = 0.23 + 0.0023 + 0.000023 + \ldots$$

$$= \frac{23}{10^2} + \frac{23}{10^4} + \frac{23}{10^6} + \ldots$$

$$= \frac{23}{10^2}\left[1 + \frac{1}{10^2} + \frac{1}{10^4} + \ldots\right]$$

$$= \frac{23}{10^2} \cdot \sum_{n=0}^{\infty} \left(\frac{1}{10^2}\right)^n = \frac{23}{100} \cdot \frac{1}{1 - \frac{1}{100}}$$

$$= \frac{23}{100} \cdot \frac{100}{99}$$

$$= \frac{23}{99}$$

(book 50)

ex: $\sum_{n=0}^{\infty} (\sqrt{2})^n$ converges or diverges?

$\sqrt{2} = 1.41\ldots > 1 \rightarrow$ the geometric series diverge.

(book 57)

ex: $\sum_{n=1}^{\infty} \frac{2}{10^n} = ?$

$$\frac{2}{10} + \frac{2}{10^2} + \ldots = \frac{2}{10}\left(1 + \frac{2}{10} + \frac{2}{10^2} \ldots\right)$$

$$= \frac{2}{10} \cdot \frac{1}{1 - \frac{1}{10}}$$

$$= \frac{2}{10} \cdot \frac{10}{9}$$

$$= \frac{2}{9}$$

$$\sum_{n=1}^{\infty} (a_n - a_{n+1}) \rightarrow \text{called telescoping series;}$$

$$\lim_{n\to\infty} (a_1 - \cancel{a_2}) + (\cancel{a_2} - \cancel{a_3}) + (\cancel{a_3} - \cancel{a_4}) + \dots + (\cancel{a_n} - a_{n+1})$$

$$= \lim_{n\to\infty} (a_1 - a_{n+1}) = a_1 - \lim_{n\to\infty} a_{n+1}$$

ex: $$\sum_{x=1}^{\infty} \left(e^{\frac{1}{x}} - e^{\frac{1}{x+1}}\right) = e^1 - \lim_{n\to\infty} e^{\frac{1}{x+1}}$$

$$= e^1 - e^0 = e - 1$$

ex: $$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = ?$$

$$\frac{1}{n(n+1)} = \frac{A}{n} + \frac{B}{n+1}$$

$$An + A + Bn = 1$$

$$A = 1 \quad B = -1$$

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1}\right)$$

$$= \left(1 - \cancel{\frac{1}{2}}\right) + \left(\cancel{\frac{1}{2}} - \cancel{\frac{1}{3}}\right) + \dots + \left(\cancel{\frac{1}{n}} - \frac{1}{n+1}\right)$$

$$= 1 - \lim_{n\to\infty} \frac{1}{n+1}$$

$$= 1 - 0 = 1$$

ex: $\sum_{n=1}^{\infty} \frac{n+1}{n} = \frac{2}{1} + \frac{3}{2} + \frac{4}{3} + \dots = \infty$

Theorem: If $\sum_{n=1}^{\infty} a_n$ converges $\lim_{n\to\infty} a_n = 0$

proof: $S_n = a_1 + \dots + a_n$

$a_n = S_n - S_{n-1}$

$\sum a_n$ converge $\Rightarrow \lim_{n\to\infty} S_n = S$

$\lim_{n\to\infty} S_{n-1} = S$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} (S_n - S_{n-1}) = \lim_{n\to\infty} S_n - \lim_{n\to\infty} S_{n-1}$$

$$= S - S = 0$$

Theorem: (n^{th} term test)

$\lim_{n\to\infty} a_n \neq 0 \Rightarrow \sum_{n=1}^{\infty} a_n$ diverges.

ex: $\sum_{n=1}^{\infty} \frac{n+1}{n}$ diverges $\lim_{n\to\infty} \frac{n+1}{n} = 1 \neq 0$

ex: $\sum_{n=1}^{\infty} n^2$ diverges since $\lim_{n\to\infty} n^2 = +\infty \neq 0$

ex: $\sum_{n=1}^{\infty} (-1)^n = -1 + 1 - 1 + 1 - 1 + \dots$ diverges

$\lim_{n\to\infty} (-1)^n$ does not exist

(book 60)

ex: $\sum_{n=1}^{\infty} \left(1-\frac{1}{n}\right)^n$ converges or diverges?

$$\lim_{n\to\infty} \left(1-\frac{x}{n}\right)^n = e^x$$

$$\lim_{n\to\infty} \left(1-\frac{1}{n}\right)^n = e^{-1} \neq 0 \Rightarrow \underbrace{\sum_{n=1}^{\infty} \left(1-\frac{1}{n}\right)^n}_{\text{diverges (by } n^{th} \text{ term test)}}$$

(book 16(a))

ex: $$\sum_{n=\infty} \frac{2^n+4^n}{3^n+4^n} = \lim_{n\to\infty} \frac{\left(\frac{2}{4}\right)^n+1}{\left(\frac{3}{4}\right)^n+1} = \frac{0+1}{0+1} = 1 \neq 0$$

by n^{th} term test the series diverges.

Theorem: $\sum_{n=1}^{\infty} a_n = A$, $\sum_{n=1}^{\infty} b_n = B$ are convergent

1) $$\sum_{n=1}^{\infty} (a_n \pm b_n) = \sum_{n=1}^{\infty} a_n \pm \sum_{n=1}^{\infty} b_n = A \pm B$$

$$(a_1+a_2+a_3)(b_1+b_2+b_3) \neq a_1b_1+a_2b_2+a_3b_3$$

$$\sum \frac{1}{n} \cdot \frac{1}{n+1} \neq \sum \frac{1}{n} \sum \frac{1}{n+1}$$

WRONG

2) $c =$ constant

$$\sum_{n=1}^{\infty} c \cdot a_n = c \sum_{n=1}^{\infty} a_n$$

ex: $\sum_{n=1}^{\infty} \frac{3^{n-1} - 1}{6^{n-1}} = ?$

$$= \sum_{n=1}^{\infty} \left(\frac{3}{6}\right)^{n-1} - \left(\frac{1}{6}\right)^{n-1}$$

$$= \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^{n} - \sum_{n=0}^{\infty} \left(\frac{1}{6}\right)^{n}$$

$$= \frac{1}{1-1/2} - \frac{1}{1-1/6} = \frac{2-6}{5} = \frac{4}{5}$$

(book 7b)

ex: For which values of x, does

$$\sum_{n=0}^{\infty} \left(-\frac{1}{2}\right)^{n} (x-3)^{n} \quad \text{converge ?}$$

$$\sum_{n=0}^{\infty} \left(\left(-\frac{1}{2}\right)(x-3)\right)^{n} = \frac{1}{1-\frac{1}{\left(-\frac{1}{2}\right)(x-3)}} = \frac{1}{1+\frac{2}{x-3}}$$

$$= \frac{x-3}{x-3+2} = \frac{x-3}{x-1}$$

if $\left|\left(-\frac{1}{2}\right)(x-3)\right| < 1 \Rightarrow |x-3| < 2 \Rightarrow 1 < x < 5$

in other words,

$$\sum_{n=0}^{\infty} \left(-\frac{1}{2}\right)^{n} (x-3)^{n} = \frac{x-3}{x-1} \quad \text{if} \quad 1 < x < 5$$

10.3. THE INTEGRAL TEST

recall: If a_n is a non-decreasing sequence bounded from above then it must converge

Suppose $\sum_{n=1}^{\infty} a_n$ is a series with $a_n \geq 0$

$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots$ either diverges to $+\infty$ or converges

harmonic series

ex: $\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \cdots$

$$\geq 1 + \underbrace{\frac{1}{2}}_{1/2} + \underbrace{\frac{1}{4} + \frac{1}{4}}_{1/2} + \underbrace{\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}}_{1/2} + \cdots + \underbrace{\frac{1}{2^{n+1}} + \cdots + \frac{1}{2^{n+1}}}_{1/2}$$

$= +\infty$

$\lim_{n\to\infty} \frac{1}{n} = 0 \Rightarrow \sum_{n=1}^{\infty} \frac{1}{n}$ diverges ex: $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges or diverges?

$\to 1 + \frac{1}{4} + \frac{1}{9} + \cdots$

$$\frac{1}{4} + \frac{1}{9} + \frac{1}{16} \leq \int_1^4 \frac{1}{x^2}\,dx$$

$$\frac{1}{4} + \cdots + \frac{1}{n^2} \leq \int_1^n \frac{1}{x^2}\,dx$$

$$1 + \frac{1}{4} + \frac{1}{9} + \cdots + \frac{1}{n^2} \leq 1 + \int_1^n \frac{1}{x^2}\,dx \leq 1 + \int_1^{\infty} \frac{1}{x^2}\,dx$$

$$= 1 + \left(-\frac{1}{x}\right)\Big|_1^{\infty}$$

$$= 1 + 1 = 2$$

$S_n \leq 2$ for all n

$\lim_{n\to\infty} S_n$ must exist since S_n is increasing and bounded from above.

✓ $a_n \geq 0$ for all n

$a_n = f(n)$

f is continuous, decreasing for $x \geq N$

Then the series $\sum_{n=N}^{\infty} a_n$ and the integral $\int_N^{\infty} f(x)\,dx$ both converge or diverge.

proof: $N=1$ case

$$\int_1^{n+1} f(x)\,dx \leq a_1 + \dots + a_n \leq a_1 + \int_1^{n} f(x)\,dx$$

$$\int_1^{\infty} f(x)\,dx \leq \sum_{n=1}^{\infty} a_n \leq a_1 + \sum_{n=1}^{\infty} a_n$$

ex: $$\int_1^{\infty} \frac{1}{x^2}\,dx \leq \sum_{n=1}^{\infty} \frac{1}{n^2} \leq 1 + \int_1^{\infty} \frac{1}{x^2}\,dx$$

$$\Rightarrow 1 \leq \sum_{n=1}^{\infty} \frac{1}{n^2} \leq 2$$

better

$$\int_2^{\infty} \frac{1}{x^2}\,dx \leq \sum_2^{\infty} \frac{1}{n^2} \leq a^2 + \int_2^{\infty} \frac{1}{x^2}\,dx$$

$$\left(\frac{-1}{x}\right)\Big|_2^{\infty}$$

$$1/2 \leq \sum_2^{\infty} \frac{1}{n^2} \leq \frac{1}{4} + \frac{1}{2}$$

$$\frac{1}{2} \leq \sum_{n=2}^{\infty} \frac{1}{n^2} \leq \frac{3}{4}$$

$$1 + 1/2 \leq 1 + \sum_{n=2}^{\infty} \frac{1}{n^2} \leq 1 + \frac{3}{4}$$

$$3/2 \leq \sum_{n=2}^{\infty} \frac{1}{n^2} + 1 \leq 7/4$$

$$\int_3^{\infty} \frac{1}{x^2}\,dx \leq \sum_3^{\infty} \frac{1}{n^2} \leq a_3 + \int_3^{\infty} \frac{1}{x^2}\,dx$$

$$1/3 \leq \sum_3^{\infty} \frac{1}{n^2} \leq \frac{1}{9} + \frac{1}{3}$$

$$1 + 1/4 + 1/3 \leq 1 + \frac{1}{4} + \sum_3^{\infty} \frac{1}{n^2} \leq 1 + \frac{1}{4} + \frac{4}{9}$$

$$19/12 \leq \sum_1^{\infty} \frac{1}{n^2} \leq \frac{61}{36}$$

exact answer $= \dfrac{\pi^2}{6}$

recall n^{th} term test

$\lim_{n\to\infty} a_n \neq 0 \Rightarrow \sum a_n$ diverges

But $\lim_{n\to\infty} a_n = 0$ does not imply $\sum a_n$ converges

ex: $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges. Let's see why!

$f(x) = \frac{1}{x}$ positive, decreasing on $x \geq 1$
continuous

$$\int_1^{\infty} \frac{1}{x}\,dx = \ln x \Big|_1^{\infty} = \infty \Rightarrow \sum_{n=1}^{\infty} \frac{1}{n} \text{ diverges}$$

✓ $$\ln(n+1) = \int_1^{n+1} \frac{1}{x}\,dx \leq 1 + \frac{1}{2} + \cdots + \frac{1}{n} \leq 1 + \int_1^{n} \frac{1}{x}\,dx = 1 + \ln n$$

$$n = e^{1000} \approx 2^{1000} \approx 10^{300}$$

$$1000 \leq 1 + \cdots + \frac{1}{e^{1000}} \leq 1 + 1000$$

** $\sum_{n=1}^{\infty} \frac{1}{n^p}$ called p series.

$p > 1$ $\quad f(x) = \frac{1}{x^p}$ decreasing, positive
continuous $[9, \infty)$

$$\int_1^{\infty} \frac{1}{x^p}\,dx = \int_1^{\infty} x^{-p}\,dx = \frac{x^{-p+1}}{-p+1}\Bigg|_1^{\infty} = \frac{0-1}{-p+1}$$

$$= \frac{-1}{-p+1} \quad (\text{since } -p+1 > 0)$$

$p=1$ $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges (Harmonic series)

$0<p<1 \Rightarrow -p+1>0$

$$\int_1^{\infty} \frac{1}{x^p}\,dx = \frac{x^{-p+1}}{-p+1} = +\infty \quad (\text{since } -p+1>0)$$

$p=0 \Rightarrow \sum_{n=1}^{\infty} \frac{1}{1} = 1+1+1+\dots = +\infty$ diverges.

$p<0$ $\sum_{n=1}^{\infty} n^{-p}$ $\lim_{n\to\infty} n^{-p} = +\infty$ so the series diverge by n^{th} test teo.

$$\boxed{\sum_{n=1}^{\infty} \frac{1}{n^p} = \begin{cases} \text{converge}, & p>1 \\ \text{diverge}, & p\leq 1 \end{cases}}$$

ex: $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ converges or diverges?

$f(x) = \frac{1}{x^2+1}$ decreasing, positive, continuous on $x \geq 1$

$$\int_1^{\infty} \frac{1}{x^2+1}\,dx = \arctan x \Big|_1^{\infty} = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4} < \infty$$

integral converge $\Rightarrow$ series converge (by integral test)

10.4. COMPARISON TESTS

Theorem: $0 \leq a_n \leq b_n$, for all n

✓ if $\sum b_n$ converges $\Rightarrow$ $\sum a_n$ converges

✓ if $\sum a_n$ diverges $\Rightarrow$ $\sum b_n$ diverges

ex: $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ converges or diverges

$$n^2+1 \geq n^2$$

$$\frac{1}{n^2+1} \leq \frac{1}{n^2} \qquad \sum_{n=1}^{\infty} \frac{1}{n^2} \text{ converges } (p=2 \text{ series})$$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^2+1}$$ converge by comparison test.

ex: $\sum_{n=1}^{\infty} \frac{5}{5n-1}$ converges or diverges ?

$$5n-1 \leq 5n \quad \text{if} \quad n \geq 1$$

$$5 \cdot \frac{1}{5n-1} \geq 5 \cdot \frac{1}{5n} = \frac{1}{n}$$

$$\sum_{n=1}^{\infty} \frac{5}{5n-1} \geq \sum_{n=1}^{\infty} \frac{1}{n} = +\infty$$

$\downarrow$

must diverge

Limit Comparison Test

Suppose $a_n > 0$, $b_n > 0$ for all n

✓ 1) if $\lim_{n\to\infty} \frac{a_n}{b_n} = c > 0$, $\underbrace{\sum_{n=1}^{\infty} a_n \text{ and } \sum_{n=1}^{\infty} b_n}$

both converge or both diverge.

2) if $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$ and $\sum b_n$ converge $\Rightarrow \sum a_n$ converge

3) if $\lim_{n\to\infty} \frac{a_n}{b_n} = +\infty$ and $\sum b_n$ diverge $\Rightarrow \sum a_n$ diverge.

proof: for all $n \geq N$ $\quad 0 < \frac{c}{2} \leq \frac{a_n}{b_n} \leq \frac{3c}{2}$

$a_n \leq \frac{3c}{2} b_n \Rightarrow$ if $\sum b_n$ converge $\sum a_n$ converge

$a_n \geq \frac{c}{2} b_n \Rightarrow$ if $\sum b_n$ diverge $\sum a_n$ diverge

ex: $\sum_{n=1}^{\infty} \frac{2n+1}{n^2+n+1}$ converges or diverges

$$a_n = \frac{2n+1}{n^2+n+1} \quad , \quad b_n = \frac{1}{n}$$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim_{n\to\infty} \frac{2n+1}{n^2+n+1}}{1/n}$$

$$= \lim_{n\to\infty} \frac{2n^2+n}{n^2+n+1} = 2 > 0$$

since $\sum \frac{1}{n}$ diverge $\Rightarrow \sum \frac{2n+1}{n^2+n+1}$ must diverge by limit comparison test.

(book 23)

ex: $\sum_{n=1}^{\infty} \frac{10n+1}{n(n+1)(n+2)}$ $\rightarrow$ $a_n \left(\approx \frac{10n}{n^3} = \frac{10}{n^2} \right)$

compare with $b_n = \frac{1}{n^2}$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = 10$$

$\sum \frac{1}{n^2}$ converge ($p=2$ series)

so $\sum \frac{10n+1}{n(n+1)(n+2)}$ must converge

(book 26)

ex: $\sum_{n=1}^{\infty} \frac{n+2^n}{n^2 2^n}$ $\rightarrow$ $a_n \approx \frac{2^n}{n^2 2^n} = \frac{1}{n^2}$

$b_n = \frac{1}{n^2}$

$$\lim_{n\to\infty} \frac{\frac{n+2^n}{n^2 2^n}}{1/n^2} \Bigg\} \lim_{n\to\infty} \frac{n^3 + n^2 2^n}{n^2 2^n} = \lim_{n\to\infty} \frac{n}{2^n} + 1 = 1 > 0.$$

l'hospital $\rightarrow 0$

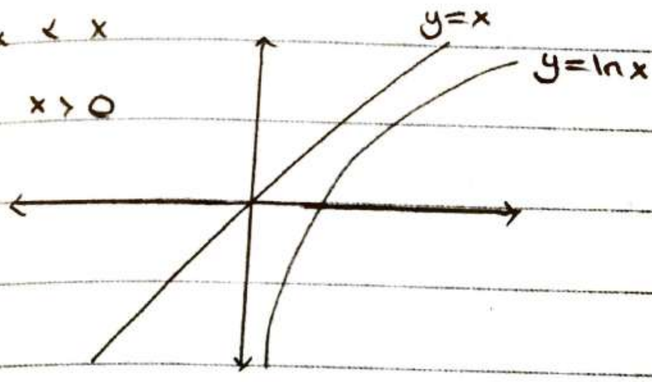
Limit comparison test says;

$\sum \frac{1}{n^2}$ and $\sum \frac{n+2^n}{n^2 2^n}$

both converge since $\sum \frac{1}{n^2}$ is converge.

$\ln x < x$

if $x > 0$



$\alpha > 0 \qquad \ln x^{\alpha} < x^{\alpha} \qquad$ if $\quad x > 0$

$$\alpha \ln x < x^{\alpha} \qquad x > 0$$

$$\ln x < \frac{1}{\alpha} \cdot x^{\alpha} \qquad x > 0$$

ex: $\ln x < 2x^{1/2} = 2\sqrt{x}$

$\ln x < 100\, x^{1/100}$

ex: $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n^{3/2}}$ $\longrightarrow n^{0.000000001}$

$$\approx \frac{n^{\epsilon}}{n^{3/2}} = \frac{1}{n^{3/2-\epsilon}}$$

$$\frac{\ln n}{n^{3/2}} \leq \frac{4n^{1/4}}{n^{3/2}} = \frac{4}{n^{3/2-1/4}} = \frac{4}{n^{5/4}}$$

$\sum \frac{4}{n^{5/4}}$ converges ($p = \frac{5}{4} > 1$ series)

by comparison test

$\sum \frac{\ln n}{n^{3/2}}$ must converge.

ex: $\sum_{n=1}^{\infty} \frac{2^n+3^n}{3^n+4^n}$ converge or diverge

$\frac{3^n}{4^n} = \left(\frac{3}{4}\right)^n$

$b_n = \left(\frac{3}{4}\right)^n$

$a_n = \frac{2^n+3^n}{3^n+4^n}$

$\sum \left(\frac{3}{4}\right)^n$ geo series convergent,

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{2^n+3^n}{3^n+4^n} \cdot \frac{4^n}{3^n}$$

$$= \lim_{n\to\infty} \frac{\left(\frac{8}{12}\right)^n + 1}{\left(\frac{9}{12}\right)^n + 1} = \frac{0+1}{0+1} = 1$$

since $0 < \text{limit} < +\infty$

and $\sum \left(\frac{3}{4}\right)^n$ converge

by comparison test $\rightarrow$ $\sum_{n=1}^{\infty} \frac{2^n+3^n}{3^n+4^n}$ must converge

10.5. ABSOLUTE CONVERGENCE THE RATIO AND ROOT TESTS

Definition: if $\sum |a_n|$ converges

then we say $\sum a_n$ converges absolutely.

Theorem: If a series converges absolutely then the series itselfs converges ex: $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n^2}$ converges or diverges ?

$\sum_{n=1}^{\infty} \left| (-1)^{n+1} \cdot \frac{1}{n^2} \right| = \sum_{n=1}^{\infty} \frac{1}{n^2}$ converges ($p=2$ series)

$\sum_{n=1}^{\infty} (-1)^{n+1} \cdot \frac{1}{n^2}$ converges absolutely hence converges

ex: $\sum_{n=1}^{\infty} \frac{\sin n}{n^2}$

WRONG $\sum_{n=1}^{\infty} \frac{\sin n}{n^2} \leq \sum_{n=1}^{\infty} \frac{1}{n^2}$

we can not say $\sum_{n=1}^{\infty} \frac{\sin n}{n}$ converges since $\sum_{n=1}^{\infty} \frac{\sin n}{n^2}$ converges. Because to use comparison test the terms of the series must be non-negative

$\left| \frac{\sin n}{n^2} \right| \leq \frac{1}{n^2}$

By comparison test $\underbrace{\sum \left| \frac{\sin n}{n^2} \right|}_{\text{must converges}} \leq \underbrace{\sum \frac{1}{n^2}}_{\text{converges}}$

must converges ⇐ converges

So $\sum \frac{\sin n}{n^2}$ converges absolutely hence must converges

The Ratio Test

let $\sum a_n$ be any series

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \rho$$

1) if $0 \le \rho < 1$ → the series converge absolutely
2) if $\rho > 1$ → the series diverges
3) $\rho = 1$ → no conclusion

ex: $\sum_{n=0}^{\infty} \frac{(2n)!}{n!.n!}$ converges or diverges?

$$\frac{a_{n+1}}{a_n} = \frac{\frac{[2(n+1)]!}{(n+1)!(n+1)!}}{\frac{2n!}{n!.n!}} = \frac{(2n+2)!}{2n!} \cdot \frac{n!.n!}{(n+1)!(n+1)!}$$

$$= \frac{(2n+2)(2n+1)\cancel{(2n!)}}{\cancel{2n!}} \cdot \frac{\cancel{n!} \cdot \cancel{n!}}{(n+1).\cancel{n!}.(n+1).\cancel{n!}}$$

$$= \frac{(2n+2)(2n+1)}{(n+1)(n+1)} \xrightarrow[n\to\infty]{} 4 > 1$$

the series diverges by ratio test.